

COHEN-MACAULAYNESS OF TRIVIAL EXTENSIONS

A. MAHDIKHANI, P. SAHANDI AND N. SHIRMOHAMMADI

ABSTRACT. Our goal is to determine when the trivial extensions of commutative rings by modules are Cohen-Macaulay in the sense of Hamilton and Marley. For this purpose, we provide a generalization of the concept of Cohen-Macaulayness of rings to modules.

1. INTRODUCTION

Throughout this paper all rings are commutative, with identity, and all modules are unital. The theory of Cohen-Macaulay rings admits a rich theory in commutative *Noetherian* rings. No attempts have been made to develop the concept of Cohen-Macaulayness to non-Noetherian rings until 1992. Then, Glaz [8] begun an investigation on the notion of Cohen-Macaulayness for non-Noetherian rings and, she asked how one can define a non-Noetherian notion of Cohen-Macaulayness such that the definition coincides with the original one in the Noetherian case, and that coherent regular rings are Cohen-Macaulay, see [9, p. 220].

More recently, Hamilton and Marley [10] established a definition of Cohen-Macaulayness for non-Noetherian rings. More precisely, employing Schenzel's weakly proregular sequences [14], they used the tool of Čech cohomology modules to define the notion of parameter sequences. A parameter sequence such that every truncation on the right is also a parameter sequence is called a strong parameter sequence. In some sense, this is a generalization of the system of parameters to the non-Noetherian case. They then called a ring *Cohen-Macaulay* if every strong parameter sequence is a regular sequence. They showed that their definition coincides with the original one in the Noetherian case and that coherent regular rings are Cohen-Macaulay (in the sense of new definition).

Let R be a ring and M be an R -module. In 1955, Nagata construct a ring extension of R called the *trivial extension* of R by M , denoted here by $R \ltimes M$. This ring is of particular importance in commutative algebra (cf. [2] and [5, Theorem 3.3.6]). One of the properties of trivial extension is as follows: if R is Noetherian local and M is finitely generated, then $R \ltimes M$ is Cohen-Macaulay if and only if R is Cohen-Macaulay and M is maximal Cohen-Macaulay, see [2, Corollary 4.14]. Motivated by this and [10, Example 4.3], we wish to investigate whether the trivial extension $R \ltimes M$ is Cohen-Macaulay with the definition of Hamilton and Marley. The main ingredient in this investigation is to formulate a definition for a module to be Cohen-Macaulay, which was left unaddressed in [10]. So we find ourselves forced to extend the notion of Cohen-Macaulay to modules.

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The outline of the paper is as follows. In Section 2, we recall some essential definitions and results on which we base our approach. In Section 3, after defining weakly proregular sequences on modules, we give a characterization of such sequences using the vanishing of suitable Čech cohomology modules. We then define (strong) parameter sequences on modules. Continually, after citing some elementary properties of such sequences, we relate these sequences to system of parameters of finitely generated modules in Noetherian local rings. In Section 4, we define Cohen-Macaulayness of modules over non-Noetherian rings. Among other things, we are able to establish our main result which says when the trivial extension $R \ltimes M$ is Cohen-Macaulay.

2. PRELIMINARIES

Let R be a ring, I be an ideal of R and M be an R -module. Following [5], a sequence $\mathbf{x} := x_1, \dots, x_\ell \in R$ is called a *weak M -regular sequence* if x_i is a non-zero-divisor on $M/(x_1, \dots, x_{i-1})M$ for $i = 1, \dots, \ell$. If, in addition, $M \neq \mathbf{x}M$, we call \mathbf{x} an *M -regular sequence*. The *classical grade* of I on M , denoted $\text{grade}(I, M)$, is defined to be the supremum of the lengths of all weak M -regular sequences contained in I , see [11].

The *polynomial grade* of I on M is defined by

$$\text{p. grade}_R(I, M) := \lim_{m \rightarrow \infty} \text{grade}(IR[t_1, \dots, t_m], R[t_1, \dots, t_m] \otimes_R M).$$

It follows from [1] and [11] that

$$\text{p. grade}_R(I, M) = \sup\{\text{grade}(IS, S \otimes_R M) \mid S \text{ is a faithfully flat } R\text{-algebra}\}.$$

Let x be an element of R . Let $C(x)$ denote the complex $0 \rightarrow R \rightarrow R_x \rightarrow 0$ where the differential is the natural localization map. For a sequence $\mathbf{x} := x_1, \dots, x_\ell$ of elements of R , the Čech complex $C(\mathbf{x})$ is inductively defined by $C(\mathbf{x}) := C(x_1, \dots, x_{\ell-1}) \otimes_R C(x_\ell)$. Then we set $C(\mathbf{x}; M) := C(\mathbf{x}) \otimes_R M$. The *i th Čech cohomology* $H_{\mathbf{x}}^i(M)$ of M with respect to the sequence \mathbf{x} is defined to be the i th cohomology of $C(\mathbf{x}; M)$.

The following summarizes some essential properties of Čech cohomology modules.

In the sequel, for a finite sequence \mathbf{x} of elements of the ring R , $\ell(\mathbf{x})$ denotes the length of \mathbf{x} .

Proposition 2.1. (see [10, Proposition 2.1]) *Let R be a ring, \mathbf{x} a finite sequence of elements of R and M an R -module.*

- (1) *If \mathbf{y} is a finite sequence of elements of R such that $\sqrt{\mathbf{y}R} = \sqrt{\mathbf{x}R}$, then $H_{\mathbf{y}}^i(M) \cong H_{\mathbf{x}}^i(M)$ for all i .*
- (2) *(Change of rings) Let $f : R \rightarrow S$ be a ring homomorphism and N an S -module. Then $H_{\mathbf{x}}^i(N) \cong H_{f(\mathbf{x})}^i(N)$ for all i .*
- (3) *(Flat base change) Let $f : R \rightarrow S$ be a flat ring homomorphism. Then $H_{\mathbf{x}}^i(M) \otimes_R S \cong H_{\mathbf{x}}^i(M \otimes_R S) \cong H_{f(\mathbf{x})}^i(M \otimes_R S)$ for all i .*
- (4) *$H_{\mathbf{x}}^{\ell(\mathbf{x})}(M) \cong H_{\mathbf{x}}^{\ell(\mathbf{x})}(R) \otimes_R M$.*

Applying parts (2) and (4) of Proposition 2.1 to the natural homomorphism $R \rightarrow R/\text{Ann } M$ together with [10, Proposition 2.7] yields the next corollary.

Corollary 2.2. *Let R be a ring and M be a finitely generated R -module of finite dimension. Then $H_{\mathbf{x}}^i(M) = 0$ for all $i > \dim M$.*

The i th *local cohomology* $H_I^i(M)$ of M with respect to I is defined by

$$H_I^i(M) := \varinjlim \operatorname{Ext}_R^i(R/I^n, M).$$

In the case that R is Noetherian, one has $H_I^i(M) = H_{\mathbf{x}}^i(M)$ for all i , where $I := \mathbf{x}R$, see [6, Theorem 5.1.20].

Let \mathfrak{p} a prime ideal of R . By $\operatorname{ht}_M \mathfrak{p}$, we mean the Krull dimension of the $R_{\mathfrak{p}}$ -module $M_{\mathfrak{p}}$. Also, for an ideal I of R

$$\operatorname{ht}_M I := \inf \{ \operatorname{ht}_M \mathfrak{p} \mid \mathfrak{p} \in \operatorname{Supp} M \cap V(I) \}.$$

3. WEAKLY PROREGULAR AND PARAMETER SEQUENCES

3.1. Weakly proregular sequences. It is mentioned in [10] that local cohomology and Čech cohomology are not in general isomorphic over non-Noetherian rings. In [14], Schenzel gave necessary and sufficient conditions on a sequence \mathbf{x} of a ring R such that the isomorphism $H_I^i(M) \cong H_{\mathbf{x}}^i(M)$ holds for all i and R -modules M , where $I := \mathbf{x}R$. Such sequences are called R -weakly proregular sequences. In the following, we provide the module theoretic version of this notion.

Let R be a ring and M be an R -module. For $x \in R$, we use $\mathbb{K}_{\bullet}(x, M)$ to denote the Koszul complex

$$0 \longrightarrow M \xrightarrow{x} M \longrightarrow 0.$$

For a sequence $\mathbf{x} = x_1, \dots, x_\ell$ the Koszul complex $\mathbb{K}_{\bullet}(\mathbf{x}, M)$ is defined to be the complex $\mathbb{K}_{\bullet}(x_1, M) \otimes_R \cdots \otimes_R \mathbb{K}_{\bullet}(x_\ell, M)$. The i th homology of $\mathbb{K}_{\bullet}(\mathbf{x}, M)$ is denoted by $H_i(\mathbf{x}, M)$ and called the Koszul homology of the sequence \mathbf{x} with coefficients in M . For $m \geq n$, there exists a chain map $\phi_n^m(\mathbf{x}, M) : \mathbb{K}_{\bullet}(\mathbf{x}^m, M) \rightarrow \mathbb{K}_{\bullet}(\mathbf{x}^n, M)$ given by

$$\phi_n^m(\mathbf{x}, M) = \phi_n^m(x_1, M) \otimes_R \cdots \otimes_R \phi_n^m(x_l, M),$$

where, for all $x \in R$, $\phi_n^m(x, M)$ is the chain map

$$\begin{array}{ccccccc} 0 & \longrightarrow & M & \xrightarrow{x^m} & M & \longrightarrow & 0 \\ & & \downarrow x^{m-n} & & \downarrow = & & \\ 0 & \longrightarrow & M & \xrightarrow{x^n} & M & \longrightarrow & 0. \end{array}$$

Hence, $\{\mathbb{K}_{\bullet}(\mathbf{x}^m, M), \phi_n^m(\mathbf{x}, M)\}$ is an inverse system of complexes. Note, for each i , the map $\phi_n^m(\mathbf{x}, M)_i$ induces a homomorphism of homology modules $H_i(\mathbf{x}^m, M) \rightarrow H_i(\mathbf{x}^n, M)$. We also denote this induced homomorphism by $\phi_n^m(\mathbf{x}, M)_i$. The sequence $\mathbf{x} = x_1, \dots, x_\ell$ is called *M -weakly proregular* if, for each n , there exists an $m \geq n$ such that the map $\phi_n^m(\mathbf{x}, M)_i : H_i(\mathbf{x}^m, M) \rightarrow H_i(\mathbf{x}^n, M)$ is zero for all $i \geq 1$ (see [14]). Note that an element $x \in R$ is *M -weakly proregular* if and only if there exists an $n \geq 1$ such that $(0 :_M x^n) = (0 :_M x^{n+1})$.

Remark 3.1. Let \mathbf{x} be a finite sequence of elements of R .

- (1) If \mathbf{x} is an M -weakly proregular sequence, then so is any permutation of \mathbf{x} .
- (2) Any M -regular sequence is M -weakly proregular.

The following result provides another description of weakly proregular sequences using Čech cohomology. Its proof is inspired by the proof of [14, Lemma 2.4]. Here, for a sequence $\mathbf{x} = x_1, \dots, x_\ell$, we use $H^i(\mathbf{x}, M)$ to denote the i th cohomology of the complex $\operatorname{Hom}_R(\mathbb{K}_{\bullet}(\mathbf{x}), M)$, where $\mathbb{K}_{\bullet}(\mathbf{x}) := \mathbb{K}_{\bullet}(\mathbf{x}, R)$, and we call it the i th Koszul

cohomology of the sequence \mathbf{x} with coefficients in M . It follows from [6, Theorem 5.2.5] that $H_{\mathbf{x}}^i(M) \cong \varinjlim H^i(\mathbf{x}^n, M)$.

Theorem 3.2. *Let \mathbf{x} be a finite sequence of elements of R . Then the following conditions are equivalent:*

- (1) \mathbf{x} is M -weakly proregular.
- (2) $H_{\mathbf{x}}^i(\text{Hom}_R(M, E)) = 0$ for all injective R -modules E and $i \neq 0$.

Proof. Assume that \mathbf{x} is M -weakly proregular and that E is an injective R -module. Then $\text{Hom}_R(\mathbb{K}_{\bullet}(\mathbf{x}^n), \text{Hom}_R(M, E)) \cong \text{Hom}_R(\mathbb{K}_{\bullet}(\mathbf{x}^n) \otimes_R M, E)$. So

$$\begin{aligned} H^i(\mathbf{x}^n, \text{Hom}_R(M, E)) &= H^i(\text{Hom}_R(\mathbb{K}_{\bullet}(\mathbf{x}^n), \text{Hom}_R(M, E))) \\ &\cong H^i(\text{Hom}_R(\mathbb{K}_{\bullet}(\mathbf{x}^n) \otimes_R M, E)) \\ &\cong \text{Hom}_R(H_i(\mathbf{x}^n, M), E) \end{aligned}$$

for all i . Hence

$$\varinjlim H^i(\mathbf{x}^n, \text{Hom}_R(M, E)) \cong \varinjlim \text{Hom}_R(H_i(\mathbf{x}^n, M), E).$$

By assumption, for all $n \in \mathbb{N}$, the homomorphism

$$\text{Hom}_R(H_i(\mathbf{x}^n, M), E) \longrightarrow \text{Hom}_R(H_i(\mathbf{x}^m, M), E)$$

is zero, for some $m \geq n$. Therefore

$$\varinjlim H^i(\mathbf{x}^n, \text{Hom}_R(M, E)) = \varinjlim \text{Hom}_R(H_i(\mathbf{x}^n, M), E) = 0.$$

So that $H_{\mathbf{x}}^i(\text{Hom}_R(M, E)) = 0$ for $i \neq 0$.

Conversely, assume that $H_{\mathbf{x}}^i(\text{Hom}_R(M, E)) = 0$ for all injective R -modules E and $i \neq 0$. Let $f : H_i(\mathbf{x}^n, M) \rightarrow E$ denote an injection of $H_i(\mathbf{x}^n, M)$ into an injective R -module E . Then $f \in \text{Hom}_R(H_i(\mathbf{x}^n, M), E) \cong H^i(\mathbf{x}^n, \text{Hom}_R(M, E))$. Since

$$\varinjlim H^i(\mathbf{x}^n, \text{Hom}_R(M, E)) = 0,$$

then there exists an $m \geq n$ such that $\text{Hom}_R(\phi_n^m(\mathbf{x}, M), 1_E) = 0$ which means $1_E f \phi_n^m(\mathbf{x}, M) = 0$. So $\phi_n^m(\mathbf{x}, M) = 0$, because f is injective. \square

The above theorem together with Proposition 2.1 immediately yields the following corollary.

Corollary 3.3. *Assume that \mathbf{x} and \mathbf{y} are finite sequences of R such that $\sqrt{\mathbf{x}R} = \sqrt{\mathbf{y}R}$. Then \mathbf{x} is M -weakly proregular if and only if \mathbf{y} is M -weakly proregular.*

As a consequence of the following theorem one obtains that any finite sequence of elements in a Noetherian ring is weakly proregular on any finitely generated module.

Theorem 3.4. *Let R be a Noetherian ring, I be an ideal of R and M be a finitely generated R -module. Then $H_I^i(\text{Hom}_R(M, E)) = 0$ for all injective R -modules E and $i \neq 0$.*

Proof. Assume that E is an injective R -module. Since the exact sequence $0 \rightarrow H_I^0(E) \rightarrow E \rightarrow E/H_I^0(E) \rightarrow 0$ is split by [6, Corollary 2.1.5], one has $E \cong H_I^0(E) \oplus E/H_I^0(E)$. So that, for all i , we have

$$H_I^i(\text{Hom}_R(M, E)) \cong H_I^i(\text{Hom}_R(M, H_I^0(E))) \oplus H_I^i(\text{Hom}_R(M, E/H_I^0(E))).$$

It is easy to see that $\text{Hom}_R(M, H_I^0(E))$ is I -torsion. Hence $H_I^i(\text{Hom}_R(M, H_I^0(E))) = 0$ for all $i \neq 0$. Since $E/H_I^0(E)$ is an injective R -module and $H_I^0(E/H_I^0(E)) = 0$,

then to complete the proof it is enough for us to show that $H_I^i(\text{Hom}_R(M, E)) = 0$ for the injective R -module E with additional condition that E is I -torsion-free. For this, let

$$\cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

be a free resolution of M . Then

$$0 \longrightarrow \text{Hom}_R(M, E) \longrightarrow \text{Hom}_R(F_0, E) \longrightarrow \text{Hom}_R(F_1, E) \longrightarrow \cdots$$

is an augmented injective resolution of $\text{Hom}_R(M, E)$ such that $H_I^0(\text{Hom}_R(F_i, E)) = 0$ for all i since $H_I^0(E) = 0$. Therefore $H_I^i(\text{Hom}_R(M, E)) = 0$ for all $i \neq 0$. \square

The following lemma will be used later.

Lemma 3.5. *Suppose that $f : R \rightarrow S$ is a flat ring homomorphism and that M is an R -module. If \mathbf{x} is M -weakly proregular, then $f(\mathbf{x})$ is $M \otimes_R S$ -weakly proregular. The converse holds if f is faithfully flat.*

Proof. This easily follows from [5, Proposition 1.6.7]. \square

3.2. Parameter sequences. Let (R, \mathfrak{m}) be a local Noetherian ring and M be a finitely generated R -module. A sequence of elements \mathbf{x} in R is said to be a *system of parameters* on M if $M/\mathbf{x}M$ has finite length and $\dim M = \ell(\mathbf{x})$. In fact, \mathbf{x} is a system of parameters on M if and only if $\text{ht}_M(\mathbf{x}R) = \ell(\mathbf{x}) = \dim M$.

Using homological properties of the rings instead of height conditions, the authors in [10] extended the notion of system of parameters in Noetherian local rings to sequences in non-Noetherian ones called strong parameter sequences. This subsection is devoted to generalize the notion of strong parameter sequences to modules.

Definition 3.6. *A finite sequence \mathbf{x} of elements of R is called a parameter sequence on M provided that the following conditions hold:*

- (1) \mathbf{x} is M -weakly proregular,
- (2) $\mathbf{x}M \neq M$,
- (3) $H_{\mathbf{x}}^{\ell(\mathbf{x})}(M)_{\mathfrak{p}} \neq 0$ for all prime ideals $\mathfrak{p} \in \text{Supp}_R(\frac{M}{\mathbf{x}M})$.

The sequence \mathbf{x} is called a *strong parameter sequence* on M if x_1, \dots, x_i is a parameter sequence on M for $i = 1, \dots, \ell(\mathbf{x})$. One may consider the empty sequence is a parameter sequence of length zero on any R -module. The empty sequence will also be considered as a regular sequence of length zero on any R -module.

Below, we state some elementary properties of parameter sequences that will be used in the course of the paper.

Proposition 3.7. *Let R be a ring and M be an R -module. Let \mathbf{x} be a finite sequence of elements of R .*

- (1) *Any permutation of a parameter sequence on M is again a parameter sequence on M .*
- (2) *Assume that $\sqrt{\mathbf{x}R} = \sqrt{\mathbf{y}R}$, $\ell(\mathbf{x}) = \ell(\mathbf{y})$, $\mathbf{x}M \neq M$ $\mathbf{y}M \neq M$ and $\text{Supp}_R(\frac{M}{\mathbf{x}M}) = \text{Supp}_R(\frac{M}{\mathbf{y}M})$. Then \mathbf{x} is a parameter sequence on M if and only if \mathbf{y} is a parameter sequence on M .*
- (3) *If $\mathfrak{p} \cdot \text{grade}_R(\mathbf{x}R, M) = \ell(\mathbf{x})$, then \mathbf{x} is a parameter sequence on M .*
- (4) *Every M -regular sequence is a strong parameter sequence on M .*
- (5) *Let $f : R \rightarrow S$ be a flat ring homomorphism. If \mathbf{x} is a (strong) parameter sequence on M and $\frac{M \otimes_R S}{f(\mathbf{x})M \otimes_R S} \neq 0$, then $f(\mathbf{x})$ is a (strong) parameter sequence on $M \otimes_R S$. The converse holds if f is faithfully flat.*

Proof. For (1) see Proposition 2.1 and Remark 3.1 and for (2) see, again, Proposition 2.1 together with Theorem 3.2.

To prove (3), we first note that $\text{p.grade}_R(\mathbf{x}^n R, M) = \text{p.grade}_R(\mathbf{x} R, M) = \ell(\mathbf{x})$ by [13, Section 5.5, Theorem 12] and that $H_i(\mathbf{x}^n, M) = 0$ for all $i \geq 1$ by [10, Proposition 2.7]. Let $\mathfrak{p} \in \text{Supp}_R(M/\mathbf{x}M)$. Hence $\mathbf{x}M_{\mathfrak{p}} \neq M_{\mathfrak{p}}$. Then $\text{p.grade}_R(\mathbf{x}R_{\mathfrak{p}}, M_{\mathfrak{p}}) < \infty$ again by [10, Proposition 2.7]. Since localization does not decrease the polynomial grade by [13, Section 5.5, Exercise 10] and the polynomial grade is bounded above by the length of the sequence (see [10, Proposition 2.7]), we see that $\text{p.grade}_R(\mathbf{x}R_{\mathfrak{p}}, M_{\mathfrak{p}}) = \ell(\mathbf{x})$ for all $\mathfrak{p} \in \text{Supp}_R(M/\mathbf{x}M)$. Hence $H_{\mathbf{x}}^{\ell}(\mathbf{x}M)_{\mathfrak{p}} \neq 0$ for all $\mathfrak{p} \in \text{Supp}_R(M/\mathbf{x}M)$ by [10, Proposition 2.7]. Therefore \mathbf{x} is a parameter sequence on M .

For (4), notice that $\ell(\mathbf{x}) \geq \text{p.grade}_R(\mathbf{x}R, M) \geq \ell(\mathbf{x})$ since $\mathbf{x}M \neq M$. Thus $\text{p.grade}_R(\mathbf{x}R, M) = \ell(\mathbf{x})$. Therefore \mathbf{x} is a strong parameter sequence on M by (3).

Finally for (5), assume that \mathbf{x} is a parameter sequence on M and $\frac{M \otimes_R S}{f(\mathbf{x})M \otimes_R S} \neq 0$. Then, by Lemma 3.5, $f(\mathbf{x})$ is $M \otimes_R S$ -weakly proregular sequence. Now, let $\mathfrak{q} \in \text{Supp}_S(\frac{M \otimes_R S}{f(\mathbf{x})M \otimes_R S})$ and set $\mathfrak{p} := f^{-1}(\mathfrak{q})$. Then $\mathfrak{p} \in \text{Supp}_R(\frac{M}{\mathbf{x}M})$ and one has the isomorphism

$$(1) \quad H_{f(\mathbf{x})}^{\ell(\mathbf{x})}(M \otimes_R S)_{\mathfrak{q}} \cong H_{\mathbf{x}}^{\ell(\mathbf{x})}(M)_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} S_{\mathfrak{q}}.$$

Since $S_{\mathfrak{q}}$ is a faithfully flat $R_{\mathfrak{p}}$ -module, one obtains that $f(\mathbf{x})$ is a parameter sequence on $M \otimes_R S$. To prove the converse, assume that f is faithfully flat. Again using Lemma 3.5, it is enough for us to show that $H_{\mathbf{x}}^{\ell(\mathbf{x})}(M)_{\mathfrak{p}} \neq 0$ for all $\mathfrak{p} \in \text{Supp}_R(\frac{M}{\mathbf{x}M})$. Assume that $\mathfrak{p} \in \text{Supp}_R(\frac{M}{\mathbf{x}M})$. Since f is flat, there exists a prime ideal \mathfrak{q} of S such that $\mathfrak{p} = f^{-1}(\mathfrak{q})$. The isomorphism

$$(\frac{M \otimes_R S}{f(\mathbf{x})M \otimes_R S})_{\mathfrak{q}} \cong (\frac{M}{\mathbf{x}M})_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} S_{\mathfrak{q}}$$

shows that $\mathfrak{q} \in \text{Supp}_S(\frac{M \otimes_R S}{f(\mathbf{x})M \otimes_R S})$. The isomorphism (1) now completes the proof. \square

Next we provide a description of parameter sequences using height condition.

Proposition 3.8. *Let R be a ring, M be a finitely generated R -module and \mathbf{x} be a finite sequence of elements of R .*

- (1) *If \mathbf{x} is a parameter sequence on M , then $\text{ht}_M(\mathbf{x}R) \geq \ell(\mathbf{x})$.*
- (2) *Further assume that R is Noetherian. Then \mathbf{x} is a parameter sequence on M if and only if $\text{ht}_M(\mathbf{x}R) = \ell(\mathbf{x})$.*

Proof. (1) If $\text{ht}_M(\mathbf{x}R) = \infty$, there is nothing to prove. So assume that $\text{ht}_M(\mathbf{x}R) < \infty$. Then there exists $\mathfrak{p} \in \text{Supp}(M) \cap V(\mathbf{x}R)$ such that $\text{ht}_M(\mathbf{x}R) = \text{ht}_M \mathfrak{p} = \dim M_{\mathfrak{p}}$. Since \mathbf{x} is a parameter sequence on M , then $H_{\mathbf{x}}^{\ell(\mathbf{x})}(M)_{\mathfrak{p}} \neq 0$. Therefore $\ell(\mathbf{x}) \leq \dim M_{\mathfrak{p}} = \text{ht}_M(\mathbf{x}R)$ by Corollary 2.2.

(2) Assume that \mathbf{x} is a parameter sequence on M . Since $\mathbf{x}M \neq M$, then $\text{ht}_M(\mathbf{x}R) < \infty$. By part (1), we have $\text{ht}_M(\mathbf{x}R) \geq \ell(\mathbf{x})$ and by Krull's Generalized Principal Ideal Theorem, we have $\text{ht}_M(\mathbf{x}R) \leq \text{ht}_R(\mathbf{x}R) \leq \ell(\mathbf{x})$. Then $\text{ht}_M(\mathbf{x}R) = \ell(\mathbf{x})$.

Conversely assume that $\text{ht}_M(\mathbf{x}R) = \ell(\mathbf{x})$. By Theorem 3.4, any sequence of elements in R is M -weakly proregular. Since $\text{ht}_M(\mathbf{x}R) = \ell(\mathbf{x}) < \infty$, then $(\mathbf{x})M \neq M$.

Let \mathfrak{p} be a minimal element of $\text{Supp}_R(M/\mathbf{x}M)$. Then \mathfrak{p} is a minimal prime ideal over $\mathbf{x}R + \text{Ann}(M)$. Hence $\mathfrak{p}/\text{Ann}(M)$ is a minimal prime ideal over $\mathbf{x}R(R/\text{Ann}(M))$ which is generated by $\ell(\mathbf{x})$ elements. Then

$$\dim M_{\mathfrak{p}} = \text{ht}_M \mathfrak{p} = \text{ht} \frac{\mathfrak{p}}{\text{Ann}(M)} = \text{ht } \mathbf{x}R \frac{R}{\text{Ann}(M)} = \ell(\mathbf{x}).$$

On the other hand, one has

$$\sqrt{\mathbf{x}R_{\mathfrak{p}} + \text{Ann}(M)R_{\mathfrak{p}}} = \sqrt{(\mathbf{x}R + \text{Ann}(M))R_{\mathfrak{p}}} = \mathfrak{p}R_{\mathfrak{p}}.$$

Hence

$$H_{\mathbf{x}R_{\mathfrak{p}}}^{\ell(\mathbf{x})}(M_{\mathfrak{p}}) = H_{\mathbf{x}R_{\mathfrak{p}} + \text{Ann}(M)R_{\mathfrak{p}}}^{\ell(\mathbf{x})}(M_{\mathfrak{p}}) = H_{\mathfrak{p}R_{\mathfrak{p}}}^{\ell(\mathbf{x})}(M_{\mathfrak{p}}) \neq 0.$$

Therefore \mathbf{x} is a parameter sequence on M . □

The Noetherian assumption in Proposition 3.8(2) is crucial. In fact, in every valuation domain of dimension 2, one can choose a weakly proregular sequence x, y such that $\text{ht}(x, y) = 2$, but x, y is not a parameter sequence, see [10, Example 3.7].

4. COHEN-MACAULAY MODULES

4.1. Definition and basic properties. In [8] and [9], Glaz raised the question that whether there exists a generalization of the notion of Cohen-Macaulayness with certain desirable properties to non-Noetherian rings. One of those is that every coherent regular ring is Cohen-Macaulay. In this direction, in [10], it is defined a notion of Cohen-Macaulayness for arbitrary commutative rings. This subsection is devoted to extend the definition of Cohen-Macaulayness for commutative rings in the sense of [10] to modules.

Definition 4.1. *An R -module M is called a Cohen-Macaulay R -module if every strong parameter sequence on M is an M -regular sequence.*

This definition agrees with the usual definition of Cohen-Macaulay finitely generated modules over Noetherian rings. Indeed, let R be a Noetherian ring and M be a finitely generated R -module. Assume that M is Cohen-Macaulay in the sense of Definition 4.1. To show that M is Cohen-Macaulay with the usual definition in the Noetherian case, it is enough to show that $\text{grade}(I, M) = \text{ht}_M I$ for all proper ideals I of R . To prove this, assume that I is a proper ideal of R and set $\text{ht}_M I = \ell$. Since $\text{ht}_M I = \text{ht } I(R/\text{Ann}(M))$, employing [5, Theorem A.2] to the ring $R/\text{Ann}(M)$ one finds the elements x_1, \dots, x_{ℓ} in I such that $\text{ht}_M(x_1, \dots, x_i) = i$ for all $i = 0, \dots, \ell$. It follows from Proposition 3.8 that x_1, \dots, x_{ℓ} is a strong parameter sequence on M . Hence it is an M -regular sequence. This yields that $\ell \leq \text{grade}(I, M) \leq \text{ht}_M I = \ell$. Therefore $\text{grade}(I, M) = \text{ht}_M I$. The converse is true by Theorem 4.2 and Proposition 4.6 below and [5, Corollary 1.6.19].

Let R be a ring and M be an R -module. If $\dim M = 0$, then M is Cohen-Macaulay. Indeed, in this situation, M has not any parameter sequences.

Thanks to polynomial grade, Koszul homology, and Čech cohomology of strong parameter sequences, our first result presents some equivalent statements of Cohen-Macaulayness. It generalizes [10, Proposition 4.2] for modules. Its proof is mutatis mutandis the same as that of [10, Proposition 4.2]. But, for the reader's convenience, we reprove it in the case of modules.

Theorem 4.2. *Let R be a ring and M be an R -module. The following conditions are equivalent:*

- (1) M is Cohen-Macaulay.
- (2) $\text{grade}(\mathbf{x}R, M) = \ell(\mathbf{x})$ for every strong parameter sequence \mathbf{x} of M .
- (3) $\text{p. grade}_R(\mathbf{x}R, M) = \ell(\mathbf{x})$ for every strong parameter sequence \mathbf{x} of M .
- (4) $H_i(\mathbf{x}, M) = 0$ for all $i \geq 1$ and for every strong parameter sequence \mathbf{x} of M .
- (5) $H_{\mathbf{x}}^i(M) = 0$ for all $i < \ell(\mathbf{x})$ and for every strong parameter sequence \mathbf{x} of M .

Proof. (1) \Rightarrow (2) Assume that \mathbf{x} is a strong parameter sequence on M ; so, by assumption, \mathbf{x} is M -regular sequence. Hence $\ell(\mathbf{x}) \leq \text{grade}(\mathbf{x}R, M)$. One also notices that $\text{grade}(\mathbf{x}R, M) \leq \text{p. grade}_R(\mathbf{x}R, M)$ by [13, Page 149] and that $\text{p. grade}_R(\mathbf{x}R, M) \leq \ell(\mathbf{x})$ by [13, Section 5.5, Theorem 13]. Therefore one has $\text{grade}(\mathbf{x}R, M) = \ell(\mathbf{x})$.

(2) \Rightarrow (3) Assume that \mathbf{x} is a strong parameter sequence on M ; so that $\ell(\mathbf{x}) = \text{grade}(\mathbf{x}R, M)$. As in the (1) \Rightarrow (2), again using [13, Page 149] and [13, Section 5.5, Theorem 13], one obtains that $\text{p. grade}_R(\mathbf{x}R, M) = \ell(\mathbf{x})$.

(3) \Rightarrow (1) Assume that \mathbf{x} is a strong parameter sequence on M . We proceed by induction on $\ell = \ell(\mathbf{x})$ to show that \mathbf{x} is M -regular sequence. If $\mathbf{x} = x_1$, then $\text{p. grade}_R(x_1R, M) = 1$ and $0 = H_1(x_1, M) = (0 :_M x_1)$. Hence x_1 is an M -regular element. Suppose that every strong parameter sequence on M of length at most $\ell - 1$ is M -regular sequence and that \mathbf{x} is a strong parameter sequence on M of length ℓ . Set $\mathbf{x}' = x_1, \dots, x_{\ell-1}$. Since \mathbf{x}' is a strong parameter sequence on M , then by hypothesis $\text{p. grade}_R(\mathbf{x}', M) = \ell - 1$. Thus, by induction, \mathbf{x}' is an M -regular sequence. Let $M' = M/\mathbf{x}'M$. Since $\text{p. grade}_R(\mathbf{x}, M) = \ell$, then $H_{\ell-i}(\mathbf{x}, M) = 0$ for all $i < \ell$ by [10, Proposition 2.7]. Hence

$$(0 :_{M'} x_\ell) = H_1(x_\ell, M') = H_1(x_\ell, M/\mathbf{x}'M) \cong H_1(\mathbf{x}, M) = 0$$

by [5, Proposition 1.6.13]. This implies that x_ℓ is an M' -regular element. Therefore \mathbf{x} is an M -regular sequence. Finally, notice that (3), (4) and (5) are equivalent by [10, Proposition 2.7]. \square

The Cohen-Macaulay property descends along faithfully flat extensions:

Proposition 4.3. *Let $f : R \rightarrow S$ be a faithfully flat ring homomorphism. Let M be an R -module. If $M \otimes_R S$ is Cohen-Macaulay S -module, then M is Cohen-Macaulay.*

Proof. Assume that $M \otimes_R S$ is Cohen-Macaulay S -module. Let \mathbf{x} be a strong parameter sequence on M . Then, by Proposition 3.7(5), $f(\mathbf{x})$ is strong parameter sequence on $M \otimes_R S$. Hence, the assumption together with Theorem 4.2, [13, Section 5.5, Theorem 19] and [10, Proposition 2.7] yields that

$$\begin{aligned} \ell(\mathbf{x}) &= \ell(f(\mathbf{x})) \\ &= \text{p. grade}_R(f(\mathbf{x})S, M \otimes_R S) \\ &= \text{p. grade}_R(\mathbf{x}R, M \otimes_R S) \\ &= \sup\{k \geq 0 \mid H_{\mathbf{x}}^i(M \otimes_R S) = 0 \text{ for all } i < k\} \\ &= \sup\{k \geq 0 \mid H_{\mathbf{x}}^i(M) \otimes_R S = 0 \text{ for all } i < k\} \\ &= \sup\{k \geq 0 \mid H_{\mathbf{x}}^i(M) = 0 \text{ for all } i < k\} \\ &= \text{p. grade}_R(\mathbf{x}R, M). \end{aligned}$$

Therefore M is Cohen-Macaulay. \square

One can immediately obtain the following corollaries.

Corollary 4.4. *Let (R, \mathfrak{m}) be a quasi-local ring and M an R -module. If $M \otimes_R \widehat{R}$ is Cohen-Macaulay \widehat{R} -module where \widehat{R} is the \mathfrak{m} -adic completion of R , then M is Cohen-Macaulay.*

Corollary 4.5. *Let R be a ring, M an R -module and t an indeterminate over R . If $M \otimes_R R[t]$ is Cohen-Macaulay $R[t]$ -module, then M is Cohen-Macaulay.*

Proposition 4.6. *Let R be a ring and M be a finitely generated R -module. If $M_{\mathfrak{m}}$ is Cohen-Macaulay $R_{\mathfrak{m}}$ -module for all maximal ideals \mathfrak{m} of R , then M is a Cohen-Macaulay R -module.*

Proof. Assume that $\mathbf{x} = x_1, \dots, x_\ell$ is a strong parameter sequence on M and that \mathfrak{m} is a maximal ideal containing $\mathbf{x}R$. Since M is finitely generated, then $M_{\mathfrak{m}} \neq (\frac{\mathbf{x}}{1})M_{\mathfrak{m}}$; so that $\frac{\mathbf{x}}{1} = \frac{x_1}{1}, \dots, \frac{x_\ell}{1}$ is a strong parameter sequence on $M_{\mathfrak{m}}$. Hence $\frac{\mathbf{x}}{1}$ is $M_{\mathfrak{m}}$ -regular sequence. Thus \mathbf{x} is M -regular sequence. Therefore M is Cohen-Macaulay. \square

As mentioned in the introduction, every coherent regular ring is Cohen-Macaulay. In particular, every valuation domain is Cohen-Macaulay. In the following, we show that every torsion-free module over such domain is Cohen-Macaulay. In fact, we do this for torsion-free modules over almost valuation domains. Recall that an integral domain R with quotient field K is called an *almost valuation domain* if for every nonzero $x \in K$, there exists an integer $n \geq 1$ such that either $x^n \in R$ or $x^{-n} \in R$ [4].

Proposition 4.7. *Every torsion-free module over an almost valuation domain is Cohen-Macaulay.*

Proof. Suppose that (R, \mathfrak{m}) is an almost valuation domain and M is a torsionfree R -module. Assume that $\mathbf{x} := x_1, x_2$ is a sequence in R of length 2. Assume that $x_1^n R \subseteq x_2^n R$ for some positive integer n . Then

$$H_{\mathbf{x}}^2(M) = H_{x_1^n, x_2^n}^2(M) = H_{x_2^n}^2(M) = 0.$$

Hence $\mathbf{x} = x_1, x_2$ can not be a parameter sequence on M . Then, for each parameter sequence \mathbf{x} of M , $\ell(\mathbf{x}) \leq 1$. Therefore M is Cohen-Macaulay. To this end, one notices that M is torsion-free. \square

Recall that the module M is called *Cohen-Macaulay in the sense of ideals* (resp. *finitely generated ideals*) if $\text{ht}_M(I) = \text{p. grade}_R(I, M)$ for all ideals (resp. finitely generated ideals) I , see [3, Definition 3.1]. The following proposition generalizes [3, Theorem 3.4] to finitely generated modules.

Proposition 4.8. *Let R be a ring and M be a finitely generated R -module. If M is Cohen-Macaulay in the sense of ideals (or finitely generated ideals), then M is Cohen-Macaulay in the sense of Definition 4.1.*

Proof. Assume that \mathbf{x} is a strong parameter sequence on M . Then, by Proposition 3.8 and [13, Section 5.5, Theorem 13], we have

$$\text{ht}_M(\mathbf{x}R) \geq \ell(\mathbf{x}) \geq \text{p. grade}_R(\mathbf{x}R, M).$$

However, by assumption $\text{p. grade}_R(\mathbf{x}R, M) = \text{ht}_M(\mathbf{x}R)$. Then $\text{p. grade}_R(\mathbf{x}R, M) = \ell(\mathbf{x})$. Therefore M is Cohen-Macaulay in the sense of Definition 4.1. \square

4.2. The Cohen-Macaulayness of some constructions. Let R and S be two commutative rings with unity, let J be an ideal of S and $f : R \rightarrow S$ be a ring homomorphism. The subring $R \bowtie^f J := \{(x, f(x) + j) | x \in R \text{ and } j \in J\}$ of $R \times S$ is called the *amalgamation of R with S along J with respect to f* [7]. This construction generalizes several classical constructions. Among them is the Nagata's trivial extension, see [7, Examples 2.5 and 2.6].

Under mild conditions, the next proposition shows that the Cohen-Macaulayness of $R \bowtie^f J$ descends to that of R .

Proposition 4.9. *Let R and S be commutative rings with unity, let J be an ideal of S and $f : R \rightarrow S$ be a ring homomorphism. Assume that J is flat as an R -module induced by f . If $R \bowtie^f J$ is Cohen-Macaulay, then R is Cohen-Macaulay. Moreover, if any strong parameter sequence on J is a strong parameter sequence on R , then J is Cohen-Macaulay.*

Proof. Note that as an R -module $R \bowtie^f J \cong R \oplus J$. This in conjunction with the assumption implies the natural embedding $\iota_R : R \rightarrow R \bowtie^f J$ is faithfully flat. Hence, by Proposition 4.3, R is Cohen-Macaulay. The rest of the conclusion is clear since J is flat. \square

We do not know whether the Cohen-Macaulay property of R ascends to that of $R \bowtie^f J$. The difficulty lies in linking strong parameter sequences of $R \bowtie^f J$ to strong parameter sequences of R and J . However, we solve this difficulty under certain assumptions. Let M be an R -module. Then $R \ltimes M$ denotes the *trivial extension* of R by M . As indicated in [7, Example 2.8], if $S := R \ltimes M$, $J := 0 \ltimes M$, and $f : R \rightarrow S$ be the natural embedding, then $R \bowtie^f J \cong R \ltimes M$.

Assume that R is Noetherian local and that M is finitely generated. It is well known that the trivial extension $R \ltimes M$ is Cohen-Macaulay if and only if R is Cohen-Macaulay and M is maximal Cohen-Macaulay, see [2, Corollary 4.14]. Our main theorem in this paper generalizes this result. To prove it we need the following lemma.

Lemma 4.10. *Let R be a ring and M be an R -module. Set $S := R \ltimes M$. Let $\pi : S \rightarrow R$ be the natural projection. Then $\mathbf{x} \subseteq S$ is S -weakly proregular if and only if $\pi(\mathbf{x})$ is R and M -weakly proregular.*

Proof. First of all notice that using the structure of prime spectrum of the trivial extension one has $\sqrt{\mathbf{x}S} = \sqrt{\pi(\mathbf{x})S}$. Using this together with Proposition 2.1(2) and Theorem 3.2, one can deduce that \mathbf{x} is S -weakly proregular if and only if $(\pi(\mathbf{x}), 0)$ is S -weakly proregular. Thus, it suffices to prove that if \mathbf{y} is a finite sequence of elements from R , then \mathbf{y} is R and M -weakly proregular if and only if $(\mathbf{y}, 0)$ is S -weakly proregular. However, as R -modules, $H_i((\mathbf{y}^n, 0), S) \cong H_i(\mathbf{y}^n, R) \oplus H_i(\mathbf{y}^n, M)$ for all i and n . Therefore the definition of weakly proregular sequence completes the proof. \square

Theorem 4.11. *Let R be a ring and M be an R -module such that every R -weakly proregular sequence is an M -weakly proregular sequence. Then $R \ltimes M$ is Cohen-Macaulay if and only if R is Cohen-Macaulay and every R -regular sequence is a weak M -regular sequence.*

Proof. First assume that $R \ltimes M$ is Cohen-Macaulay and that \mathbf{x} is a strong parameter sequence on R . So, in particular, \mathbf{x} is M -weakly proregular. By Lemma 4.10,

$(\mathbf{x}, 0) \subseteq R \ltimes M$ is $R \ltimes M$ -weakly proregular. Also, one notices that

$$\frac{R \ltimes M}{(\mathbf{x}, 0)R \ltimes M} = \frac{R \ltimes M}{\mathbf{x}(R \ltimes M)} \cong \frac{R}{\mathbf{x}R} \ltimes \frac{M}{\mathbf{x}M} \neq 0.$$

Finally, for a prime ideal $\mathfrak{p} \ltimes M$ of $R \ltimes M$ containing $(\mathbf{x}, 0)$, using Proposition 2.1 and [6, Exercise 6.2.12(iv)], one has

$$H_{(\mathbf{x}, 0)}^{\ell(\mathbf{x})}(R \ltimes M)_{\mathfrak{p} \ltimes M} \cong H_{\mathbf{x}}^{\ell(\mathbf{x})}(R)_{\mathfrak{p}} \oplus H_{\mathbf{x}}^{\ell(\mathbf{x})}(M)_{\mathfrak{p}} \neq 0.$$

Therefore $(\mathbf{x}, 0)$ is a strong parameter sequence on $R \ltimes M$. So that $(\mathbf{x}, 0)$ is a $R \ltimes M$ -regular sequence. It easily follows that \mathbf{x} is R -regular sequence and that \mathbf{x} is weak M -regular sequence. This shows that R is Cohen-Macaulay and every R -regular sequence is a weak M -regular sequence.

Conversely assume that R is Cohen-Macaulay and every R -regular sequence is a weak M -regular sequence. Let \mathbf{x} be a strong parameter sequence on $R \ltimes M$. Then, using Lemma 4.10, $\pi(\mathbf{x}) \subseteq R$ is R and M -weakly proregular sequence. Also notice that

$$R \ltimes M \neq \sqrt{\mathbf{x}(R \ltimes M)} = \sqrt{\pi(\mathbf{x})(R \ltimes M)} = \sqrt{\pi(\mathbf{x})R} \ltimes M.$$

Hence $R \neq \pi(\mathbf{x})R$. Now let \mathfrak{p} be a prime ideal of R containing $\pi(\mathbf{x})$. So that the prime ideal $\mathfrak{p} \ltimes M$ of $R \ltimes M$ contains \mathbf{x} . Hence $H_{\mathbf{x}}^{\ell(\mathbf{x})}(R \ltimes M)_{\mathfrak{p} \ltimes M} \neq 0$. On the other hand, again using Proposition 2.1 and [6, Exercise 6.2.12(iv)], one has

$$H_{\mathbf{x}}^{\ell(\mathbf{x})}(R \ltimes M)_{\mathfrak{p} \ltimes M} \cong H_{\pi(\mathbf{x})}^{\ell(\mathbf{x})}(R \ltimes M)_{\mathfrak{p} \ltimes M} \cong H_{\pi(\mathbf{x})}^{\ell(\mathbf{x})}(R)_{\mathfrak{p}} \oplus H_{\pi(\mathbf{x})}^{\ell(\mathbf{x})}(M)_{\mathfrak{p}}.$$

Consequently, Proposition 2.1(4) yields that $H_{\pi(\mathbf{x})}^{\ell(\mathbf{x})}(R)_{\mathfrak{p}} \neq 0$. This means that $\pi(\mathbf{x})$ is a strong parameter sequence on R ; hence $\pi(\mathbf{x})$ is R -regular sequence and so by our assumption $\pi(\mathbf{x})$ is a weak M -regular sequence. Then $H_{\pi(\mathbf{x})}^i(R) = 0$ for all $i < \ell(\mathbf{x})$ by Theorem 4.2. Note that $\pi(\mathbf{x})^n$ is a weak M -regular sequence for all positive integer n ([5, Exercise 1.1.10]), we have $H_{\ell(\mathbf{x})-i}(\pi(\mathbf{x})^n, M) = 0$ for all $i < \ell(\mathbf{x})$ ([5, Theorem 1.6.16]). Hence in view of [5, Proposition 1.6.10(d)] $H_{\pi(\mathbf{x})}^i(M) = \varinjlim H^i(\pi(\mathbf{x})^n, M) = 0$ for all $i < \ell(\mathbf{x})$. Therefore

$$H_{\mathbf{x}}^i(R \ltimes M) \cong H_{\pi(\mathbf{x})}^i(R \ltimes M) \cong H_{\pi(\mathbf{x})}^i(R) \oplus H_{\pi(\mathbf{x})}^i(M) = 0$$

for all $i < \ell(\mathbf{x})$. It now follows from Theorem 4.2 that $R \ltimes M$ is Cohen-Macaulay. \square

We note that the backward direction in Theorem 4.11 holds in general, without supposing that every R -weakly proregular sequence is an M -weakly proregular sequence.

Let R be a ring and M be an R -module. It can be seen from the proof of the above theorem that $R \ltimes M$ is Cohen-Macaulay if and only if every strong parameter sequence on $R \ltimes M$ of the form $(\mathbf{x}, 0)$ is a regular sequence.

Corollary 4.12. (c.f. [2, Corollary 4.14]) *Let R be a Noetherian local ring and M be a finitely generated nonzero R -module. The following statements are equivalent:*

- (1) $R \ltimes M$ is Cohen-Macaulay;
- (2) R is Cohen-Macaulay and every R -regular sequence is an M -regular sequence;
- (3) R is Cohen-Macaulay and M is maximal Cohen-Macaulay.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TABRIZ, TABRIZ, IRAN
E-mail address: a.mahdikhani@tabrizu.ac.ir

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TABRIZ, TABRIZ, IRAN
E-mail address: sahandi@ipm.ir

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TABRIZ, TABRIZ, IRAN.
E-mail address: shirmohammadi@tabrizu.ac.ir